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On Solutions of Nonlinear Wave Equations

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Abstract

It is shown that for certain nonlinear functions $f(u)$ solutions of the Cauchy problem for the equation

$$u_{tt} - c^2 \Delta u = f(u)$$

become infinite at a finite value of t. The number of space dimensions is one, two or three and the initial data must satisfy appropriate conditions. A similar result is established for the nonlinear Euler-Poisson-Darboux equation

$$u_{tt} + \frac{k}{t} u_t - c^2 \Delta u = f(u)$$

for two space variables and $k > 1$. The results are based upon a comparison theorem which enables us to compare different solutions of the same or of related equations.

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1. Introduction

Consider the nonlinear wave equation

$$(1) \qquad u_{tt} - c^2 \Delta u = f(u), \qquad f \in C^{(2)}$$

For $t \geq 0$ let u be a solution of (1) which satisfies the initial conditions

$$(2) \qquad u(x,0) = \phi(x), \qquad \phi \in C^{(2)}$$

$$(3) \qquad u_t(x,0) = \Psi(x), \qquad \Psi \in C^{(2)}.$$

We will show that for a certain class of functions f(u) the solution u becomes infinite at a finite value of t, provided the initial data satisfy appropriate conditions. In this demonstration the number of space dimensions n is restricted to be one, two or three. The same conclusion will also be deduced for solutions of the nonlinear Euler-Poisson-Darboux equation

$$(4) \qquad u_{tt} + \frac{k}{t} u_t - c^2 \Delta u = f(u), \qquad f \in C^{(2)}$$

for $n = 2$, $k > 1$ and $\Psi = 0$.

The proof will be based upon a comparison theorem which permits us to compare certain pairs of solutions u and u' having different initial data and/or different functions f(u). If u' is independent of x, it can be calculated explicitly and provides an explicit bound on u. For certain f(u) the explicit lower bound shows that u becomes infinite at a finite value of t which can be estimated. This procedure is similar to that which we have used to treat corresponding problems for certain nonlinear elliptic equations[1].

After the completion of the present work an abstract by P.C. Rosenbloom[2] was published in which the same result was stated for (1) with $n = 1$ and $f(u) = u^2$.

It was also stated that the result holds for a class of hyperbolic equations which was not specified. However I have been told by Prof. Rosenbloom that his method is the same as the one used here. Therefore I believe that his results, which are still unpublished, must coincide with mine.

2. Theorems and proofs

We begin by writing the solution of (1)-(3) treating f(u) as the inhomogeneous term in the linear wave equation. The solution formulas for that case now become the following nonlinear integral equations for n = 1, 2 and 3 respectively:

$$(5)\quad u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(\xi)d\xi + \frac{1}{2}\Big[\phi(x+ct) + \phi(x-ct)\Big] + \frac{1}{2c^2}\int_0^{ct}\int_{x-\rho}^{x+\rho} f\Big[u(\xi,t-c^{-1}\rho)\Big]d\xi d\rho,$$

$$(6)\quad u(x,t) = \frac{1}{2\pi c}\iint_{r\leq ct}\frac{\Psi(\xi,\eta)d\xi d\eta}{\sqrt{c^2t^2-r^2}} + \frac{\partial}{\partial t}\,\frac{1}{2\pi c}\iint_{r\leq ct}\frac{\phi(\xi,\eta)d\xi d\eta}{\sqrt{c^2t^2-r^2}}$$

$$+\frac{1}{2\pi c^2}\int_0^{ct} d\rho \iint_{r\leq\rho}\frac{f[u(\xi,\eta,t-c^{-1}\rho)]d\xi d\eta}{\sqrt{\rho^2-r^2}},$$

$$(7)\quad u(x,t) = \frac{t}{4\pi}\iint_{\Omega}\Psi(x+\alpha ct)d\omega + \frac{\partial}{\partial t}\frac{t}{4\pi}\iint_{\Omega}\phi(x+\alpha ct)d\omega + \frac{1}{4\pi c^2}\int_0^{ct}\rho d\rho\iint_{\Omega} f\Big[u(x+\alpha\rho,t-c^{-1}\rho)\Big]d\omega$$

In (6), r is defined by $r^2 = (x-\xi)^2 + (y-\eta)^2$, while in (7) α denotes a unit vector and $d\omega$ denotes the area element on the unit sphere Ω.

The integral equations (5)-(7) can be rewritten in a simpler form if we denote by $u_o(x,t)$ the first two terms on the right-hand side of each equation and by L f(u) the last term. Then (5)-(7) become

$$u = u_o + L\, f(u). \tag{8}$$

The function u_o is just the solution of the initial-value problem when $f \equiv 0$, and the operator L is a linear integral operator with a positive kernel. The domain of integration in L is that part of the retrograde characteristic cone with vertex at x,t which lies in the half-space $t \geq 0$. We will denote this closed portion of the cone by $C(x,t)$.

Now we can prove the

> Comparison theorem. Let u be a solution of (1)-(3) in one, two or three space dimensions and let u' be another solution with f', $\emptyset'$, Ψ' in place of f, $\emptyset$, Ψ. Suppose that $f(u) \geq f'(u')$ whenever $u \geq u'$. If $u_o > u_o'$ in $C(x_o,t_o)$ then $u > u'$ in $C(x_o,t_o)$. If $u_o \geq u_o'$ in $C(x_o,t_o)$ and f' is Lipschitz continuous, then $u \geq u'$ in $C(x_o,t_o)$.

Proof. The solution u satisfies (8) and u' satisfies

$$u' = u_o' + L\, f'(u'). \tag{9}$$

First suppose $u_o > u_o'$ in $C(x_o,t_o)$. Let t_1 be the least value of t at which $u' = u$ in $C(x_o,t_o)$ and let x_1 be a point in $C(x_o,t_o)$ at which $u'(x_1,t_1) = u(x_1,t_1)$. Then, upon using (8) and (9) at x_1,t_1 and subtracting, we obtain

$$0 = u_o(x_1,t_1) - u_o'(x_1,t_1) + L\big[f(u) - f'(u')\big]. \tag{10}$$

The expression $u_o(x_1,t_1) - u_o'(x_1,t_1)$ in (10) is positive by hypothesis. The term $L\big[f(u) - f'(u')\big]$ is non-negative. To see this, note that the integral in L extends over the cone $C(x_1,t_1)$ which lies within $C(x_o,t_o)$ and in the half-space $t < t_1$. But in this region $u > u'$, from the definition of t_1 and from the fact that initially $u = u_o > u_o' = u'$. Thus the right-hand side of (10) is the sum of a positi and a non-negative term and cannot be zero. This contradiction shows that t_1 does not exist. Since initially $u > u'$, this inequality holds throughout

$C(x_o,t_o)$. Thus the first half of the theorem is proved.

To prove the second half of the theorem, we denote by t_1 the greatest lower bound of the values of t for which $u(x,t) < u'(x,t)$ in $C(x_o,t_o)$. Let x_1, $t_1 + \varepsilon$, with ε sufficiently small, be a point in $C(x_o,t_o)$ at which $u(x_1,t_1+\varepsilon) < u'(x_1,t_1+\varepsilon)$. Then from (8) and (9) we obtain instead of (10)

$$(11)\quad u(x_1,t_1+\varepsilon) - u'(x_1,t_1+\varepsilon) = u_o - u_o' + L\left[f(u) - f'(u')\right].$$

The term $u_o - u_o'$ is non-negative by assumption. The integral operator L may be split up into an integral from $t = 0$ to $t = t_1$, and one from $t = t_1$, to $t = t_1+\varepsilon$. The first integral is non-negative by the definition of t_1 and the properties of f, f' and L. The second integral may again be split into two parts, one over the region where $u \geq u'$ and the other over the remainder. To estimate the last integral we note that

$$(12)\quad f(u) - f'(u') = \left[f(u) - f'(u)\right] + \left[f'(u) - f'(u')\right].$$

The first term on the right is non-negative by hypothesis. Thus we can finally rewrite (10) as

$$(13)\quad u-u' = b + \overset{t_1+\varepsilon}{\underset{t_1}{L}} \left[f'(u) - f'(u')\right].$$

Here $b \geq 0$ denotes the sum of the four non-negative terms which we have considered on the right side of (11) and ${}^{t_1+\varepsilon}L_{t_1}$ denotes the integral operator extended from $t = t_1$ to $t = t_1+\varepsilon$ over that part of the region where $u < u'$. Since f' is Lipschitz continuous, we may estimate the integral by

$$(14)\quad \overset{t_1+\varepsilon}{\underset{t_1}{L}} \left[f'(u) - f'(u')\right] \geq \varepsilon K(u-u')_{min}.$$

Here K is a positive constant proportional to the Lipschitz constant of f' and

$(u-u')_{min}$ is the minimum value of $u-u'$ in the retrograde cone $C(x_1,t_1+\varepsilon)$, $t \geq t_1$. We now choose ε so small that $\varepsilon K < 1$ and then apply (13) to that point in $C(x_1,t_1+\varepsilon)$ at which $u-u'$ achieves its minimum. Then we obtain

(15) $$(u-u')_{min} \geq b + \varepsilon K(u-u')_{min}.$$

Thus

(16) $$(u-u')_{min} \geq \frac{b}{1-\varepsilon K} \geq 0.$$

This consequence is contrary to the assumption that $u-u'$ is negative in $C(x_1,t_1+\varepsilon)$. Thus t_1 does not exist and the second half of the theorem is proved.

It is to be observed that the comparison theorem also holds for solutions of the more general Cauchy problem in which Cauchy data are prescribed on some space-like surface S, rather than on $t = 0$. In this case (8) still applies, the domain of integration in L being that portion of the retrograde cone lying above S (i.e. on the same side of S as the vertex). The solution u_0 is given by formulas similar to the first two terms in (5)-(7).

An immediate consequence of the comparison theorem is

Corollary I. If for $|x-x_0| \leq a$ the initial data satisfy

$n=1$: $\phi(x) \geq \phi'(x)$, $\Psi(x) \geq \Psi'(x)$

$n=2,3$: $\phi(x) = \phi'(x)$, $\Psi(x) \geq \Psi'(x)$.

then $u(x,t) \geq u'(x,t)$ in $C(x_0,a/c)$.

Proof. We use the above hypotheses in the explicit expressions for u_0 which are given in (5)-(7). We then see that $u_0(x,t) \geq u_0'(x,t)$ for x,t in $C(x_0,a/c)$. The corollary then follows from the comparison theorem.

To apply the corollary, suppose that $\phi'(x) = \alpha$, $\Psi'(x) = \beta$ for $|x-x_0| \leq a$, where α and β are constants. Then we can find a solution $u'(x,t) = u'(t)$ which is independent of x in $C(x_0,a/c)$. For this solution (1)-(3) become

(17) $$u'_{tt} = f'(u'),$$

(18) $$u'(0) = \alpha$$

(19) $$u'_t(0) = \beta.$$

Thus in this case $u(x,t)$ is bounded below in $C(x_0,a/c)$ by $u'(t)$. If the inequalities in Corollary I and in the Comparison Theorem were all reversed, or if primed and unprimed quantities were interchanged, then $u(x,t)$ would be bounded above by $u'(t)$.

This application of Corollary I can be summarized in

Corollary II. If for $|x-x_0| \leq a$ the initial data satisfy

$n=1$: $\phi(x) \geq \alpha$, $\Psi(x) \geq \beta$

$n=2,3$: $\phi(x) = \alpha$, $\Psi(x) \geq \beta$

then $u(x,t) \geq u'(t)$ in $C(x_0,a/c)$, where $u'(t)$ satisfies (17)-(19). If the inequalities in the hypotheses are reversed then the inequality in the conclusion is also reversed, that is, $u(x,t) \leq u'(t)$.

To obtain the explicit solution for $u'(t)$ let us first consider the case

(20) $$\beta \geq 0, \ \beta^2 + \int_\alpha^z f'(s)ds > 0 \text{ for } z > \alpha.$$

Then the solution is given by

(21) $$t = \int_\alpha^{u'} \left[\beta^2 + \int_\alpha^z f'(s)ds\right]^{-1/2} dz, \qquad t \geq 0.$$

If the integral in (21) converges to a finite value T as $u' \to \infty$, then u' becomes infinite at $t = T$, where

(22) $$T = \int_\alpha^{\infty} \left[\beta^2 + \int_\alpha^z f'(s)ds\right]^{-1/2} dz.$$

When the hypotheses of Corollary II are satisfied, the conclusion $u \geq u'$

shows that u also becomes infinite at some value of $t \leq T$, provided $T \leq a/c$. Furthermore, u becomes infinite in $C(x_0,T)$, as we see by replacing a by cT in the corollary.

Suppose that instead of (20) we have the case

$$\beta < 0, \qquad \beta^2 + \int_\alpha^z f'(s)ds > 0 \quad \text{for } z > u'_m. \tag{23}$$

Here u'_m is the largest root ($<\alpha$) of the equation

$$\beta^2 + \int_\alpha^{u'_m} f'(s)ds = 0, \qquad u'_m < \alpha. \tag{24}$$

Then the solution is given by

$$t = -\int_\alpha^{u'} \left[\beta^2 + \int_\alpha^z f'(s)ds\right]^{-1/2} dz, \qquad 0 \leq t \leq t_m, \tag{25}$$

$$t = t_m + \int_{u'_m}^{u'} \left[\beta^2 + \int_\alpha^z f'(s)ds\right]^{-1/2} dz, \qquad t \geq t_m. \tag{26}$$

The quantity t_m is defined by

$$t_m = -\int_\alpha^{u'_m} \left[\beta^2 + \int_\alpha^z f'(s)ds\right]^{-1/2} dz. \tag{27}$$

As before u' becomes infinite at $t = T$ if the integral in (26) converges to the value T, and then, by Corollary II, u also becomes infinite if $T \leq a/c$. Here T is given by

$$T = t_m + \int_{u'_m}^{\infty} \left[\beta^2 + \int_\alpha^z f'(s)ds\right]^{-1/2} dz. \tag{28}$$

We summarize the above results in

Theorem 1: Let u be a solution of (1)-(3). Suppose there exists a Lipschitz continuous function $f'(u')$ and two constants α and β such that

$$\begin{aligned} &f(u) \geq f'(u') \quad \text{for } u \geq u', \\ &n=1: \quad \emptyset(x) \geq \alpha, \quad \Psi(x) \geq \beta \quad \text{for } |x-x_0| \leq cT, \\ &n=2,3: \quad \emptyset(x) = \alpha, \quad \Psi(x) \geq \beta \quad \text{for } |x-x_0| \leq cT, \end{aligned}$$

and such that (20) is satisfied and the integral in (22) converges or (23), (24) are satisfied and the integral in (28) converges. Then u becomes infinite in $C(x_0,T)$.

Here T is given by (22) or (28) according as $\beta \geq 0$ or $\beta < 0$.

Remarks:

1. If f' is positive and $\Psi(x) \geq \beta = 0$ then (20) is satisfied.

2. If $f' = O(u^{1+\varepsilon})$, $\varepsilon > 0$ as $u \to +\infty$ then the integrals in (22) and (28) converge.

3. Suppose that $n = 1$, that (20) is satisfied, that the integral in (22) converges and that $\emptyset(x) \geq \alpha$ for all x. Then the condition $\emptyset(x) \geq \alpha$ for $|x-x_0| \leq cT$ is satisfied and the theorem applies. Only if $\emptyset(x) \to -\infty$ is it possible that this condition be violated, in which case the theorem would not apply.

4. Since T increases as α and β decrease, the best result (i.e. least value of T) is obtained when the largest possible values of α and β are used.

5. If $f \geq 0$ it follows from the comparison theorem that $u \geq u_0$. By a proof similar to the proof of that theorem it follows that if $u_0 \geq m$ in $C(x_0,t_0)$ and if $f(u) \geq 0$ for $u \geq m$ then $u \geq u_0$ in $C(x_0,t_0)$. Consequently if $f'(u') \geq 0$ for $u' \geq m'$ and if $u' \geq m'$ in $C(x_0,t_0)$ it suffices for the above theorem that $f(u) \geq f'(u')$ for $u \geq u' \geq m'$. If $\alpha \geq m'$ then $u' \geq m'$ in $C(x_0,t_0)$.

6. If $f'(u')$ is non-decreasing the condition $f(u) \geq f'(u')$ for $u \geq u'$ follows from the simpler condition $f(u) \geq f'(u)$.

3. The nonlinear Euler-Poisson-Darboux equation

Let us now consider a solution u of the nonlinear equation (4) which satisfies the initial conditions (2) and (3). We will deal with two space dimensions (n=2) and suppose that $k > 1$ in (4) and $\Psi \equiv 0$ in (3). Then, as in the previous case, the solution formula for the linear equation yields a nonlinear integral equation for u. The solution formula, which has been determined by J.B. Diaz and G.S.S. Ludford, is (cf. [3], equations (23),(41). The sign of their integral is incorrect in (37) and (41).)

$$(29) \qquad u(x,t) = u_o(x,t) + \frac{1}{2\pi}\int_0^t dt_o \iint\limits_{\alpha^2+\beta^2\leq 1} V(\delta,t,t_o) f\Big[u(x+\alpha c[t-t_o],t_o)\Big] d\alpha d\beta .$$

In (29) u_o is the solution of the initial-value problem when $f \equiv 0$, α denotes the vector (α,β), $\delta^2 = \alpha^2 + \beta^2$ and V is given by

$$(30) \qquad V(\delta,t,t_o) = \frac{2^k\, t_o^k(t-t_o)}{(1-\delta^2)^{1/2}\Big[(t+t_o)^2-\delta^2(t-t_o)^2\Big]^{k/2}} F\left(\frac{k-1}{2}, \frac{k}{2}, \frac{1}{2}, z\right).$$

Here F is the hypergeometric function and

$$(31) \qquad z = \frac{1-\delta^2}{\left(\frac{t+t_o}{t-t_o}\right) - \delta^2} .$$

Since $0 \leq t_o \leq t$ and $0 \leq \delta^2 \leq 1$ it follows that $0 \leq z \leq 1$. For this range of z and for $k > 1$ the hypergeometric function F in (30) is non-negative. This follows at once from the series for F[4, p.56, (2)]. Therefore V is also non-negative and the integral operator in (29) has a non-negative kernel. Since (29) can be written in the form (8) with a non-negative integral operator L, and the integration extends over the same cone C(x,t) as in the previous case, the comparison theorem again applies. In the statement of this theorem for solutions of (2)-(4) with $n=2$ and $k > 1$ it must be specified that $\Psi(x) \equiv \Psi'(x) \equiv 0$.

The solution u_o of the initial-value problem when $f \equiv 0$ was found by A. Weinstein[5]. It is

$$(32) \qquad u_o(x,t) = \frac{\omega_{k-1}}{\omega_{k+1}} \iint\limits_{\alpha^2+\beta^2 \leq 1} \phi(x+\alpha ct, y+\beta ct)(1-\alpha^2-\beta^2)^{\frac{k-3}{2}} \, d\alpha \, d\beta,$$

where

$$(33) \qquad \omega_m = 2\pi^{m/2} / \Gamma\left(\tfrac{m}{2}\right).$$

From (32) it follows that if $\phi(x) \geq \phi'(x)$ for $|x-x_o| \leq a$ then $u_o \geq u_o'$ in $C(x_o, a/c)$. Furthermore if $\phi'(x) = \alpha$ for $|x-x_o| \leq a$ then in $C(x_o, a/c)$ both u_o' and u' are functions of t only. The equation (4) satisfied by $u'(t)$ in $C(x_o, a/c)$ then becomes

$$(34) \qquad u'_{tt} + \frac{k}{t} u'_t = f'(u').$$

The initial conditions (2),(3) are

$$(35) \qquad u'(0) = \alpha, \quad u'_t(0) = 0.$$

Upon combining these results with the comparison theorem we have the

> *Corollary III.* Let u be a solution of (2)-(4) with $n=2$, $k > 1$ and $\Psi \equiv 0$, $\phi(x) \geq \alpha$ for $|x-x_o| \leq a$. Let $f'(u')$ be a Lipschitz continuous function such that $f(u) \geq f'(u')$ whenever $u \geq u'$. Then $u(x,t) \geq u'(t)$ in $C(x_o, a/c)$ where $u'(t)$ is the solution of (34) and (35).

We have previously shown[1] that the solution u' of (34) and (35) becomes infinite at a finite value of t - say T_o - provided that $f'(u') > 0$ and that the integral in (22) converges. If $T_o \leq a/c$ then the solution u considered in Corollary III also becomes infinite in $C(x_o, T)$. If f' is nondecreasing then T_o can be estimated by

$$(36) \qquad T \leq T_0 \leq \sqrt{k+1}\, T.$$

Here T is given by (22) with $\beta = 0$.

We may summarize this result in

Theorem 2. Let u be a solution of (2)-(4) with $n=2$, $k > 1$, and $\Psi(x) \equiv 0$, $\phi(x) \geq \alpha$ for $|x-x_0| \leq a$. Let $f'(u')$ be a positive Lipschitz continuous function such that $f(u) \geq f'(u')$ whenever $u \geq u'$ and such that the integral in (22) converges. If $T_0 \leq a/c$ then $u(x,t)$ becomes infinite in $C(x_0,T_0)$.

The remarks following Theorem 1 also apply here.

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